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## NOTE ON ISOGONAL TRANSFORMATION; PARTICULARLY ON OBTAINING CERTAIN SYSTEMS OF CURVES WHICH OCCUR IN THE STATICS OF POLYNOMIALS.\*

By DR. ROLLIN A. HARRIS, Washington, D. C.

1. If  $s, = \xi + i\eta$ , be a function  $\chi$  of  $u, = X + iY$ , then if

$$f(\xi, \eta) = 0 \quad (1)$$

be the equation of any path of  $s$ ,

$$f(\text{r}\chi(u), \text{p}\chi(u)) = 0$$

is the equation of its image ( $\text{r}\chi$  denoting the real part of  $\chi$  and  $\text{p}\chi$  the imaginary part).

2. If  $u$  be a function of  $z, = x + iy$ ,

$$f(\text{r}\chi(u), \text{p}\chi(u)) = 0$$

may be interpreted upon either the  $u$ -plane or the  $z$ -plane; that is, it may be regarded as an equation in  $X, Y$  or in  $x, y$ .

3. We may substitute for  $\chi(u)$  its derivative with respect to  $z$  (or  $x$ ), and the result will still be a function of  $z$ , and so, of course, of  $u$ .

The accent will be used to denote a differentiation with respect to  $x$ . When the quantity to which it is applied is a function of  $z$ , the accent may also be regarded as denoting a differentiation with respect to that letter.

4. If two curves

$$f_1(\xi, \eta) = 0, \quad f_2(\xi, \eta) = 0 \quad (2) \quad (3)$$

intersect at certain angles in the  $s$ -plane,

$$\begin{aligned} f_1'(\text{r}\chi(u), \text{p}\chi(u)) &= 0, \\ f_2'(\text{r}\chi(u), \text{p}\chi(u)) &= 0 \end{aligned}$$

\* An elegant exposition of the subject here referred to may be found in an article by Félix Lucas entitled *Statique des polynômes*. Bulletin de la Société Mathématique de France, 1889.

See also the same author's memoirs in *Comptes Rendus*, especially 1868 (with a review upon them in 1870) and 1888.

intersect at like angles in the  $u$ -plane (or  $z$ -plane), provided  $ds/du$  (or  $ds/dz$ ) neither vanish nor become infinite at the points of intersection of (2) and (3).

5.  $s = u$ .

The images of the orthogonal systems

$$\xi^2 + \eta^2 = (\text{constant})^2, \quad (4)$$

$$\eta/\xi = \text{constant} \quad (5)$$

are the orthogonal systems

$$X^2 + Y^2 = (\text{constant})^2, \quad (6)$$

$$Y/X = \text{constant}. \quad (7)$$

In the  $z$ -plane,\* supposing  $u$  to be a polynomial in  $z$  of order  $m$ , equation (6) denotes a system of curves of order  $2m$  which have been called *lemniscates of the  $m$ th order*, or *Cassinoids*, since the continued product of the moduli of the factors of  $u$  is constant for any given curve of the system. Equation (7) denotes a system of curves of order  $m$ , each having  $m$  hyperbolic branches setting out from the zeros of  $u$ , and have been called *hyperbolas of the  $m$ th order*. All asymptotes pass through the centre of mean position of the zeros, thus dividing the  $z$ -plane into  $2m$  equal angular compartments; for this reason the curves have been also called *stelloids*.

6. Regarding the zeros of the polynomial  $u$  as fixed points of unit mass which act upon the variable point inversely as their distances from it, equation (6) represents a system of *equipotential lines* in the  $z$ -plane. Upon the same suppositions, equation (7) represents a system of *lines of force*.

$$7. s = \frac{\partial^n u}{\partial x^n}.$$

The image of any curve

$$f(\xi, \eta) = 0$$

is

$$f\left[\frac{\partial^n X}{\partial x^n}, \frac{\partial^n Y}{\partial x^n}\right] = 0. \quad (8)$$

In particular, this shows that because the systems (4) and (5) are orthogonal, so are the systems

$$\left[\frac{\partial^n X}{\partial x^n}\right]^2 + \left[\frac{\partial^n Y}{\partial x^n}\right]^2 = (\text{constant})^2, \quad (9)$$

$$\frac{\partial^n Y}{\partial x^n} / \frac{\partial^n X}{\partial x^n} = \text{constant}. \quad (10)$$

\* Journal de l'École Polytechnique, 1879: Géométrie des polynômes.

Holzmillér, Theorie der isogonalen Verwandtschaften, S. 202, 204 \*, 205 \*\*).

Annals of Mathematics, Vol. iv, p. 73.

If  $n = 1$ , the curves (9) become

$$X'^2 + Y'^2 = (\text{constant})^2, \quad (11)$$

which are *lines of equal expansion* for the function  $u$ ; the orthogonal trajectories are

$$Y'/X' = \text{constant}. \quad (12)$$

$$8. \quad s = \frac{\partial^n \log u}{\partial x^n}.$$

The image of any curve

$$f(\xi, \eta) = 0$$

is

$$f\left[\frac{\partial^n \log R}{\partial x^n}, \frac{\partial^n \theta}{\partial x^n}\right] = 0. \quad (13)$$

When  $n = 1$ ,  $s$  becomes  $u'/u$ , and (13)

$$f(\log' R, \theta') = 0.$$

Equation (4) now transforms into

$$\frac{X'^2 + Y'^2}{X^2 + Y^2} = (\text{constant})^2. \quad (14)$$

These curves are *isodynamic lines*, upon the hypotheses made in § 6.  $\xi, -\eta$  are equal to the total component forces along the  $x$ - and the  $y$ -axis respectively.

From equation (5) it follows that the system

$$\frac{XY' - X'Y}{XX' + YY'} = \text{constant} \quad (15)$$

is orthogonal to (14). These curves are *lines of parallel action*.

Lucas's generalization of Rolle's theorem. Since

$$\xi + i\eta = \frac{u'}{u}$$

it is clear that a variable point in the  $z$ -plane will be in equilibrium when, and only when, it coincides with a zero of the derived equation  $u' = 0$ .

*Every closed convex contour surrounding the roots of an algebraic equation surrounds also the roots of the derived equation.*

9. *Application to Hydrodynamics.* A steady irrotational motion is supposed to take place in planes parallel to  $xy$ .

Let  $X = \text{constant}$  be the lines of equal velocity-potential; then  $Y = \text{constant}$  are the stream-lines, (11) the lines of equal velocity, (12) the lines along which the direction of flow is constant.

Here and in § 7, the only condition imposed upon  $u$  is that it be a function of  $z$ ; it may, therefore, be replaced by any given function of  $u$ , and the four systems of curves just mentioned will be altered accordingly. If  $\log u$  be substituted for  $u$ , these systems assume the forms (6), (7), (14), and (15), respectively. Suppose, in a fluid of unlimited extent, line sources, each of the same strength and perpendicular to the plane  $xy$ , to pass through the zeros of a polynomial  $u$ . For this motion (6) are the lines of equal velocity-potential, (7) the stream-lines, (14) the lines of equal velocity, and (15) the lines along which the direction of flow is constant.